Boolean Circuits

Monotone Circuits

Lemma
$f_t(G) = 1 \wedge c_t(G) = 0$
$\Rightarrow \exists i \ : \ f_i = f_{i_1} \wedge f_{i_2}, (c_{i_1} \sqcap c_{i_2})(G) < (c_{i_1} \wedge c_{i_2})(G)$

Theorem (Valiant)
$D_{M_2}(\text{Majority}_n) \leq 5.3 \cdot \log n$

Theorem (lower bound for $\text{Clique}_3(n)$)

$C_{M_2}(\text{Clique}_3(n)) = \Omega(n^3/\log^4 n)$

Claim
$(c_{i_1} \sqcup c_{i_2})(G) > (c_{i_1} \vee c_{i_2})(G)$ for at most 2^{n-p} negative testgraphs.

Corollary
Every monotone circuit for $\text{Clique}_3(n)$ contains at least $\binom{n}{3}/m^2$ And-gates or 2^{p-1} Or-gates.

Claim
$(c_{i_1} \sqcap c_{i_2})(G) < (c_{i_1} \wedge c_{i_2})(G)$ for at most m^2 positive testgraphs.

Lemma
Any set of more than $m = 2(p-1)^2$ edges either contains a matching of size p or a star of size p.

Theorem (Razbarov)

$$C_{M_2}(\text{Clique}_k(n)) \geq \left(\frac{n}{k^3 \log^2 n}\right)^{\lfloor \sqrt{k} \rfloor}$$

for $k = \left(\frac{n}{2\log^2 n}\right)^{\frac{1}{3}}$.

Theorem
For all $f \in M_n$ holds
$C_{M_2}(f) = \mathcal{O}(2^n \cdot n^{-3/2} \cdot \log n)$.

Lemma
If $L(f) > 1$, then for all $\rho \in R^n_{n-1}$ holds
$E(L_{U_2}(f_\rho)) \leq (1 - 1.5/n) L_{U_2}(f)$.

Corollary
For all $\rho \in R^n_{n-1}$,
$E(L_{U_2}(f_\rho) \leq (1 - 1.5/n)(L_{U_2}(f) - 1/3)$

Theorem (Subbotovskaya)
Let f be a formula. Then
$E(L_{U_2}(f_\rho) \leq \varepsilon^{1.5}(L_{U_2}(f) - 1/3)$
for all $\rho \in R^n_{\varepsilon n}$.

Theorem
For almost all $f \in M_n$
- $C_{M_2}(f) \geq c_1 \cdot 2^n \cdot n^{-3/2}$
- $L_{M_2}(f) \geq c_2 \cdot 2^n \cdot n^{-1/2} \cdot \log^{-1} n$
- $D_{M_2}(f) \geq n - 1/2 \log n - \log\log n + \log c_2$

Lemma (random partial assignments)
For all $\rho \in R^n_{n-1}$, $E(L(f_\rho)) \leq (1 - 1/n) L(f)$

Formula Complexity

Theorem
There is a formula $f \in B_n$ with $L_{U_2}(f) \geq 2^n/\log n$.

Corollary

$$E_{\rho \in R^n_m}\left(L_{U_2}(\text{An}^*_\rho)\right) \geq \frac{1}{2} \cdot \frac{n}{\log\log n}$$

Theorem (Andreev)
$L_{U_2}(\text{An}) = \Omega(n^{2.5}/((\log n)^{1.5}(\log\log n)^{2.5}))$

Lemma
For all $\rho \in R^n_m$ with $m := 2\log n \log\log n$,

$$\Pr[\bigwedge_{i=1}^{\log n} \#\rho|^{-1}_{x_i}(*) \geq 1] \geq 1/2.$$

Lemma
For all $\rho \in R^n_{\log n}$,

$$\forall 1 \leq i \leq \log n, \ \#\rho|^{-1}_{x_i}(*) \geq 1$$
$$\Longrightarrow \ L_{U_2}(\text{An}^*_\rho) \geq n/\log\log n$$

Unbounded fan-in circuits

Proposition
For all probability distributions ω on $\{0,1\}^n$ and all $\delta > 0$ exists a multilinear polynomial $p \in \text{GF}_3[X_1, \ldots, X_n]$ with $\deg(p) \geq 2\lceil \log 1/\delta \rceil$ s.t.

$$\Pr[L(x) \neq p(x)] \leq \delta \quad (L \in \{\text{AND}, \text{OR}\}).$$

Lemma
$\Pr[f(x) \neq p(x)] \leq \delta$ for all $f \in B_n$, $d := D(f)$, $\varepsilon > 0$, $p \in \text{GF}_3[X_1, \ldots, X_n]$ with $\deg(p) \geq (2\lceil \log s/\varepsilon \rceil)^d$.

Theorem (Razbarov/Smolenski)
For sufficiently large n, each unbounded fan-in circuit with depth d for Parity_n has size at least

$$0.075 \cdot (\sqrt{2})^{n^{1/(2d)}}.$$

Proposition
For all $p \in \text{GF}_3[X_1, \ldots, X_n]$ with $\deg(p) \leq \sqrt{n}$,

$$\Pr_{x \in \{0,1\}^n}[p(x) \neq \text{Parity}_n(x)] \geq 0.15.$$

Proposition
For all $p : \{-1,1\}^n \to \{-1,1\} \in \text{GF}_3[X_1, \ldots, X_n]$ with $\deg(p) \leq \sqrt{n}$,

$$\Pr_{x \in \{-1,1\}^n}[p(x) = \text{Parity}^*_n(x)] \leq 0.85.$$

Ring Sum Expansion
Each boolean function f has the unique representation

$$f(x) = \bigoplus_{A \subseteq \{1,\ldots,n\}} a_A \wedge \bigwedge_{i \in A} x_i.$$

Upper bound (Lupanov)
For all $f \in B_n$, $C_{U_2}(f) = (1 + o(1))2^n/n$.

Lower bound (Shannon)
$\exists f \in B_n \ : \ C_{B_2}(f) \geq 2^n/n$.